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ABSTRACT

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SENSE AND REFERENCE IN THE DESIGN OF INTERACTIVE ILLUSTRATIONS FOR RATIONAL NUMBERS

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Abstract

It is a common tactic in the teaching of arithmetic to provide the learner with pictures and embodiments of various kinds. However, the pedagogy of illustrations is not well understood. Analysis of the concept of meaning in relation to arithmetic shows that full understanding of arithmetic implies intellectual possession of a number of different intellectual constructions, among them a teleological, an analytical, and a referential semantics for arithmetic. It is argued that the primary purpose of illustrations for elementary arithmetic is to clarify the semantics of the language of arithmetic. An informal analytic semantics for rational number concepts is presented, revealing a rather complex system of concepts. Construction of a referential semantics for the concept of ratio leads to the conjecture that illustrations for rational numbers are necessarily incomplete in the sense that they can, in principle, only illustrate some aspects of rational numbers but not others. The results of these analyses are summarized in a prescriptive theory of illustration design.

Introduction

Mathematics is the intellectually most straining subject taught in elementary school. The difficulty of mathematics has several sources. However, the intuition of educators tend to single out the ***abstractness*** of mathematics as the major source of difficulty. The abstract character of mathematics prevents the learner from "seeing" the subject matter clearly, or - to change metaphor - from "grasping" it firmly. As a result, mathematical procedures are not understood, even when they are carried out correctly. The lack of understanding - so the argument continues - causes incorrect application of mathematical concepts and procedures in the solving of novel problems.

The remedy, then, is to concretize mathematics. There are two closely related ways in which teachers, inspired by the two metaphors of "seeing" and "grasping", have tried to achieve this. One way of providing concreteness is to use ***pictures***, drawings or diagrams which visually display mathematical objects or operations. In this approach, mathematical objects and operations acquire meaning by becoming visible. Displaying a function by drawing its graph is perhaps the most common use of a picture in mathematics. However, many other mathematical objects, properties, and operations have been pictorially explained in mathematics education, although the pictures do not always have as clear a rationale as in the case of function graphs. Arnheim (1969) and in particular Sawyer (1964) provide interesting collections of examples. As a glance at any textbook in elementary mathematics will show (e.g., Rucker & Dilley, 1981), the use of pictures is a very popular teaching tactic. However, there is little research on how and why and when pictures help learning; recent reviews of research into mathematics education hardly mention the issue (e.g., Davis, 1984; Romberg & Carpenter, 1986).

A related, but distinct approach to concretizing mathematics is to use manipulatives or ***embodiments***, physical objects which can be manipulated in ways which are analogous to certain mathematical operations. Embodiments are visible, of course, but their purpose is to be objects

of actions. In this approach, mathematical objects become meaningful by becoming touchable. Embodiments are probably used less frequently in teaching than pictures, but there exists a small body of - somewhat discouraging - research on their effectiveness (Bell, Costello, & Kuchermann, 1983, pp. 190-194).

Computer technology and advanced programming techniques make it possible to construct teaching materials which combine the two approaches to concreteness. We can provide the learner with pictures on the computer screen ***and*** a set of commands for acting upon them, e. g., changing them, moving them about, combining them, etc. I will use the term ***interactive illustrations*** for graphical displays which are responsive to the learner. Computer graphics allows us to design the visual qualities of an illustration as well as the structure of the actions associated with it so as to optimize its explanatory effect, thus helping the learner to "grasp" the mathematical objects and to "see" the laws which regulate their behavior.

The new possibilities created by computer graphics raises the question of how we can design interactive illustrations for mathematics in a ***principled*** manner. What are the key features of illustrations? What are the main issues in the design of illustrations? How should one proceed in designing an illustration for a particular mathematical concept? What decisions have to be made, and on what grounds should they be made? This chapter is an attempt to provide the foundation for a discussion of such questions with respect to a particular instructional topic. The main achievement of the present chapter is a prescriptive theory for the design of illustrations for arithmetic.

Since illustrations are intended to increase the understanding of arithmetic, a theory of illustration design has to take into account what it means to understand arithmetic. A secondary achievement of this chapter is a synthesis of different approaches to the concept of understanding with respect to arithmetic. It turns out that "complete" understanding of arithmetic implies

intellectual possession of a number of different intellectual constructions. The fundamental issue in the design of an illustration is which of these intellectual constructions the illustration is intended to help the learner achieve.

The theory proposed in this chapter is developed in the context of designing an illustration for the arithmetic of rational numbers. Rational numbers are notoriously difficult to understand for primary and middle school learners. One type of evidence for this is the errors children make (Evertz, 1982; Tatsouka, 1984). A second type of evidence is provided by the curriculum. If we look at the standard school curriculum, we see that rational numbers and related issues take up approximately 20 lessons spread out over 3 or more years. To learn the arithmetic of rational numbers is a major effort. Also, it is an area of arithmetic in which pictures and embodiments are frequently used. The tertiary achievement of the chapter is the design of a novel illustration for this topic.

The chapter is divided into five sections. The first section below locates the use of illustrations within the effort of teaching arithmetic with understanding. I argue that the main function of an illustration is to provide ***referential*** meaning, i. e., that illustration design has to begin with semantic, rather than procedural, considerations, a view not generally held in the mathematics education community. The second section presents some preliminary results with respect to an analytical semantics for rational number concepts. The main conclusion is that there is a family of such concepts and that the appropriateness of a particular illustration is a function of which concept is employed. Consequently, a particular concept, a sense, has to be selected before a referential semantics - an illustration - can be designed. The third section gives an example of the design of a referential semantics for the ratio concept. Evaluation of the example illustration leads to the conjecture that illustrations for rational numbers are necessarily incomplete in the sense that each illustration can illuminate some aspects of rational numbers but not others. These analyses are brought together in a fourth, theoretical section on the design of

interactive illustrations for arithmetic. Finally, the fifth section contains a brief discussion of some issues not treated in the body of the chapter.

Meaning in Arithmetic

Everyone agrees that instruction should make arithmetic "meaningful". However, the concept of "meaning" is hardly transparent when applied to language, and it becomes even less so when applied to arithmetic. Different perspectives generate different conceptions of what it is to "understand arithmetic". Since illustrations are supposed to help the learner understand arithmetic, principled design of illustrations requires a clear notion of what kind of understanding is involved. I will discuss arithmetic from three perspectives as ***theory***, as ***activity***, and as ***language***.

Arithmetic as Theory

Mathematical knowledge is organized into theories: the theory of equations, the theory of continuous functions, the theory of matrices, etc. The content of each theory consists of mathematical laws or principles which describe the nature and properties of the mathematical involved. Thinking of arithmetic in this way leads one to locate the meaning of arithmetic in its laws or principles. The basic arithmetic functions of addition, subtraction, multiplication, and division are governed by algebraic laws, such as closure, associativity, commutativity, laws of distribution, laws of identity elements, and of inverse. See Figure 1 for examples. Part of knowing and understanding arithmetic is to know and understand those laws. They constitute a theory of numbers, or, more accurately, a theory of the number system.

However, the hypothesis that understanding of arithmetic consists of knowledge of those laws is implausible. Certainly human beings can perform, say, multi-column subtraction with understanding without being able to state the properties of the number system which are

exemplified in Figure 1. Indeed, to most of us, the algebraic laws of the number system seem to *require* explanation (e. g., why is addition commutative, while division is not?), rather than *constitute* an explanation of arithmetic. Laws like those shown in Figure 1 represent the output from the understanding process, not the input into it.

The Theorem: *There is just one arithmetic operation* **A** *such that, for any whole numbers* x *and* y,

$$\mathbf{A}(x, 0) = x$$

and

$$\mathbf{A}(x, successor(y)) = successor(\mathbf{A}(x, y))$$

This operation is known as the sum of x and y, and is usually written "$x + y$".

Theorem: $(x + y) + z = x + (y + z)$ *for every* x, y, *and* z.

Theorem: $x + y = y + x$ *for every* x *and* y.

Theorem: *If* $y + x = z + x$, *then* $y = z$.

Theorem: *If* x *and* y *are any whole numbers, then either* $x = u + y$ *for some* u, *or* $y = v + x$ *for some* v.

Figure 1. Examples of algebric laws of the number system adapted with minor changes in notation from Theorems 3-8 by Thurstone (1956, pp. 51-53).

Historically, the discovery of the algebraic structure of the number system came after the discovery of the four basic arithmetic functions and the procedures for computing them. The study of algebraic structures was undertaken in order to construct a systematic development of the number system (see, e.g., Thurston, 1956), an effort that blends into abstract algebra on the one hand and into number theory on the other. Explicit statement of the laws of the number system was an advanced mathematical achievement.

For present purposes, the implication of this is that the ***primary*** purpose of illustrations for arithmetic cannot be to clarify the laws of the number system. If understanding of arithmetic precedes knowledge of the theory of numbers, then the illustrations used to promote initial understanding must already have done their work - so to speak - when the learner confronts that theory. A closely related purpose of illustrations is to clarify why the arithmetic laws are true, i. e., to give intuitively convincing demonstrations, in lieu of the strict proofs which presumably should take their place when the learner has reached a more mature age. But understanding is not the same as believing. Most young learners are willing to take for granted the truth of the mathematical laws presented to them in schools - very sensibly so. Their difficulties do not reside in the justification of the algebraic laws, but in the content or substance of those laws. These considerations imply that illustrations which provide insights into or justifications for the laws of the number system are not of central importance to initial instruction in arithmetic.

In summary, although the algebraic laws of the arithmetic functions from a mathematical point of view constitute the content of arithmetic, intellectual possession of those laws requires understanding of arithmetic, rather than the other way around. Thus, the primary purpose of illustrations in the initial teaching of arithmetic is not to communicate those laws.

Arithmetic as Activity

Arithmetic is not just the study of the properties of the basic arithmetic functions, but also the study of how they can be computed. The purpose of teaching arithmetic is not only to enable the learner to understand the nature of numbers, but also to enable him/her to ***find*** the value of an arithmetic function for any given pair of numbers. (This emphasis on computation is virtually absent from other, higher areas of mathematics, in which the main activity is to conjecture and prove general properties of functions or of classes of functions.) Arithmetic is inherently constructive, in the sense of that term used by intuitionist mathematicians (Calder, 1979),[1] i.e., in arithmetic, mathematical objects (the sums, differences, products, and quotients of pairs of numbers) are produced, rather than just being talked about in the abstract.

What is the relationship between the algebraic laws of the number system, and the computational procedures by which the values of the four basic arithmetic functions are computed? The laws of arithmetic constitute what in computer science is known as an ***abstract specification*** of the four basic arithmetic functions. ***Any*** procedure which obeys the algebraic specification of, say, addition is a correct addition procedure; indeed, that is the only precise definition of "correct addition procedure".

However, in pedagogy, the basic arithmetic functions are not introduced as computational procedures which fulfill certain specifications. They are introduced on the bases of the semantics of those functions, e. g., addition is introduced as the mathematical description of actions of joining and combining, division as cutting into parts, etc. Thus, in pedagogy, the intellectual relationship between the algebraic laws of arithmetic and the computational procedures is reversed: Once the procedures are introduced, they can be investigated in order to ***discover*** that

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[1] An area of mathematics which is similar to arithmetic in this respect is the domain of constructions in plane geometry. Construction tasks have the general form "Given a figure F, construct a figure G with properties so-and-so!" where the construction consists of a sequence of actions with a specified set of tools. The reason why geometric constructions never became a central topic in mathematical research is perhaps that the constructions so far have resisted systematization and algorithmization.

they do, in fact, obey those laws. In this way of proceding, arithmetic has to be understood before those laws can be discovered, in accordance with the conclusion of the previous section that the laws do not themselves represent understanding of arithmetic. In short, when arithmetic is viewed as activity, understanding arithmetic means understanding its computational procedures, rather than understanding the laws of the number system.

This in turn raises the question of what it means to understand a procedure. What is the difference between blind and insightful execution of a procedure? This question has lead some researchers to look for ***the principles which underly the computational procedures*** of arithmetic. It is an interesting and startling fact that the principles which are embodied in the arithmetic procedures are not the principles of arithmetic as mathematicians understand them. The algebraic laws which hold for the arithmetic functions do not suffice for computing the values of those functions (given certain arguments). One can not derive the ***value*** of, say, (5 + 6) from the knowledge that addition is commutative, obeys the law of associativity, etc. The value has to be constructed with help of some computational procedure and the principles underlying that procedure are not identical to the algebraic laws of the number system.

The one-one principle.

"To follow this principle, a child has to coordinate two component processes, ***partitioning*** and ***tagging***. By partitioning we mean the step-by-step maintenance of two categoriers of items - those that are to be counted ... and those that have already been counted ... . Items must be transferred (either mentally or physically) one at a time from the to-be-tagged category to the already tagged category. Coordinated with this process is a second process that involves summoning up one at a time distinct tags (numerons). ... As an item is transferred from the to-be-counted category to the counted category, a distinct tag must be withdrawn from the set of mental tags. ... In other words, the one-one-principle in counting requires the rhythmic

coordination of the partitioning and tagging processes. The two processes must start together, stop together, and stay in phase throughout their use." (Gelman & Gallistel, 1978, pp. 78-79.)

The stable-order principle.

"The tags (numerons) [a child] uses to correspond to items in an array must be arranged or chosen in a stable - that is, a repeatable - order." (Gelman & Gallistel, 1978, pp. 79.)

The cardinal principle.

"The cardinal principle says that the final tag in [a counting series] has a special significance. This tag, unlike any of the preceeding tags, represents a property of the set as a whole." (Gelman & Gallistel, 1978, pp. 79-80.)

The abstraction principle.

"The abstraction principle says that the preceeding principles can be applied to ***any*** array or collection of entities." (Gelman & Gallistel, 1978, pp. 80.)

The order-irrelevance principle.

"This principle says that the order of enumeration is irrelevant; that the order in which the items are tagged, and hence which item receives which tag, is irrelevant." (Gelman & Gallistel, 1978, pp. 82.)

Figure 2. The principles underlying the procedure for counting a set of objects according to Gelman and Gallistel (1978).

For instance, the principles underlying our common procedures for counting have been analyzed by Gelman and Gallistel (1978). They are shown in Figure 2. As a second example, Resnick and

Omanson (in press) has provided an analysis of the principles underlying multi-column subtraction; they are shown in Figure 3.

Additive composition of quantities.

Any quantity can be expressed as a sum of other quantities, usually in many different ways. A quantity can be "taken apart" into a set of components, and then "put together" again without any change in its value.

Conventions of decimal place value notation.

In a decimal place value notation, each position in a multidigit number represents a successively higher power of ten. The digit in the rightmost position represents the quantity (1 * d), where d is the value of the digit; the digit in the rightmost position but one represents (10 * d); etc.

Calculation through partitioning.

An arithmetic operation of multi-digit numbers can be carried out by decomposing the numbers into components, operating upon the components, and then constructing the answer to the original computation by combining the results of the partial computations. For instance, this is the principle that permits written subtraction to be done column by column.

Recomposition and conservation of the minuend quantity.

According to this principle, it is permitted in a multi-column subtraction problem to decompose the minuend and recompose it again in any way, as long as its value is not changed. Such recomposition of the minuend will not change the answer to the subtraction problem.

Figure 3. The principles underlying written subtraction according to Resnick and Omanson (in press).

What kind of principles are these, and in what sense do they underlie the computational algorithms of counting and subtraction, respectively? Inspection of Figures 2 and 3 shows that these principles do not deal with arithmetic *per se*, but with what computer scientists would call the *data-structures* of arithmetic, e. g., place-value notation, and with what computer scientists would call issues of *control*, i. e., decisions about when to do what. The kind of principles uncovered by Gelman and Gallistel (1978) and by Resnick and Omanson (in press) apply to *the activity of computing* rather than to *the mathematical entities* involved in the computation. I will call principles of this kind *operational principles*, to distinguish them from algebraic principles.

Operational principles contribute to the understanding of arithmetic by providing *justification*; the principles just mentioned explain why the computational procedures of counting and subtraction are the way they are. Intellectual possession of these principles allows the learner to *explain* a procedure, why it works, and why it is allowed. The notions of justification and explanation have been given formal expression by VanLehn and Brown (1980) in their study of so-called planning nets. A planning net is a method for deriving a procedure from (a) its goal, (b) the constraints it has to obey, and (c) the elementary operations which can appear in it. This formalism provides a precise definition of what it means for a principle or set of principles to justify a (step in) a procedure. VanLehn and Brown (1980) call this type of analysis *teleological semantics*.

One implication of the planning net analysis is that a test for the understanding of a procedure should reveal whether the procedure can be changed appropriately when the circumstances under which it is executed are changed. This idea has been applied by Greeno, Riley, and Gelman (1984) in a theoretical study of counting procedures. They used the principles of Gelman and Gallistel (1978) in a planning net analysis of the type proposed by VanLehn and Brown (1980). The result is a computer system which can plan a counting procedure, based on

the principles of counting, given a specific counting-task. The program is able to act as if it understood the concept of counting.

Should we conceive of illustrations primarily as tools for communicating arithmetic procedures? The discussion of embodiments and pictures in the literature about arithmetic education has often assumed that the purpose of an illustration is to provide meaning for the computational procedures. Seeing the procedures in operation in a medium that is visually richer than the symbolic language of arithmetic is supposed to make them meaningful. This is the essential idea behind, for instance, Dienes blocks, which were intended to provide a concrete version of the procedure for, say, multi-column subtraction.

However, there are several reasons to doubt the pedagogical efficiency of this kind of illustration. First, careful analysis of the specific steps taken when doing multi-column subtraction with numbers and with Dienes blocks reveals that the procedures are not isomorphic. Trading a "tens block" against ten "unit blocks" is not similar to putting a scratch mark over a number and writing the result of subtracting unity from that number above it. The relationship between these two steps is, in fact, obscure enough so that one needs to understand subtraction in order to see the isomorphism, contrary to the pedagogical purpose of producing understanding with the help of the isomorphism. Second, the idea of teaching procedures with the help of embodiments relies on an (usually unstated) assumption that transfer of training will occur from the procedure as practiced to the target procedure. Given the difficulty of transferring solutions from one problem to another even under very favourable conditions, it is unclear why we should expect such transfer to take place. Third, even if the isomorphism could be improved, it is not clear why we should expect a procedure to be easier to learn with respect to concrete objects than with respect to symbols; indeed, if the isomorphism were perfect, i. e., if the procedures were to correspond exactly, then we ought to expect them to be equally difficult to learn. For these reasons, I conjecture that illustrations intended to facilitate procedure acquisition are not useful

in elementary mathematics education.

One might object that we should focus on the operational principles, instead of on the procedures themselves. After all, the purpose of an illustration is to facilitate understanding, and the understanding of a procedure, I have just argued, resides in its teleological semantics, i. e ., the operational principles and the connections between those principles and the particular steps in the procedure. However, operational principles refer to the activity of computing, rather than to the objects involved in the computation. It is unclear what it means to illustrate such principles. One suggestion would be to design a medium in which the (analogues of) the operational principles cannot be violated, so that incorrect procedural steps cannot be taken. One might object to this suggestion, in turn, by asking why such an illustration would teach anyone to avoid incorrect steps in a medium where those steps *can* be taken. Perhaps other, pedagogically useful ways can be found to illustrate operational principles. But at the present time, we lack any compelling examples of such illustrations.

In summary, there is a difference between understanding the number system and understanding the computational procedures of arithmetic. In both cases, understanding can be thought of as being in intellectual possession of a set of principles. However, the principles involved are different in the two cases. The operational principles which underlie the computational procedures are usually not stated explicitly in mathematical works. They resemble the specifications of data-structures and control-structures of computer programs. Understanding a procedure requires some general scheme for relating procedures and operational principles. Such a scheme is known as a teleological semantics. There are several reasons to believe that illustrations which intend to clarify arithmetic procedures and their associated teleological semantics are not of central importance for initial instruction in arithmetic.

Arithmetic as Language

Arithmetic is carried out in a special-purpose symbolism which allows arithmetic statements to be written succinctly, problems to be stated elegantly, and algorithms to be carried out in a convenient manner. To understand arithmetic is certainly to understand those symbols and the expressions in which they appear. The symbols constitute a part of our language, although a highly refined and specialized part. Arithmetic symbols are linguistic devices for talking about a particular subject matter, namely numbers and quantities. Hogben (1936, Chap. 3) makes the point that numbers act like nouns, the arithmetic operations act like verbs, and equations like statements.

If arithmetic symbolism is a language, then we might clarify the notion of understanding arithmetic by looking at what it means to understand a (natural) language. Studies of the semantics of natural languages have lead to the distinction between ***sense*** and ***reference***. The referent of an expression is the object in the real world which the expression talks about or stands for or asserts something about. The sense of an expression is the idea or concept which lies behind it. (Further discussion of this distinction can be found in linguistic textbooks, e.g., Lyons, 1969, pp. 424-434).

Applying this distinction to arithmetic, we can ask what sense we can assign to the arithmetic symbols and expressions. Some recent research on elementary arithmetic has shown that this question is more complicated than anticipated. The mathematical operations of adding and subtracting can have several different senses, depending upon context (Carpenter & Moser, 1982, 1983; Nesher, 1982; Vergnaud, 1983; Riley, Greeno, & Heller, 1983). For instance, "to add" can have the sense "to join" or "to put together", or, more generally, "to combine". On the other hand, it can also have the sense "to increase", or, more generally, "to change". In the same way, the subtraction symbol can have the sense "to take away" or "to decrease".

Depending upon which sense is being assigned to an arithmetic operation, different senses are appropriate for the numbers involved in the operation. For instance, when using the "change" interpretation of addition, it is natural to assign the first addend the sense of "base" or "measure of interest", while the second addend is assigned the sense of "amount of change". In a "combine" interpretation of addition, it is more natural to assign symmetrical senses to the two addends, e. g., the sense of "parts".

Being able to assign the right sense to the mathematical symbols in the right context is an important part of understanding arithmetic. This aspect of arithmetic is not thoroughly analyzed in the literature on mathematics education. One possible reason for this is that we do not yet have good analyses of the possible senses of various arithmetic concepts. In the next section, I will use some techniques from the study of semantics in order to be begin such an analysis for the concept of rational numbers.

Looking at the other component of the semantics of a language, reference, we can ask what we refer to with the symbols in an arithmetic expression. This question is complicated by the abstract nature of arithmetic. From a mathematical point of view, number symbols refer to ***numbers***, ideal objects which have a controversial kind of existence (see Benacerraf & Putnam, 1983). Thus, "8" refers to a mathematical object, namely the number eight. However, in the application of arithmetic to real world situations, we let "8" refer to, say, the length of a rectangle which is 8 units long. The ability to make such assignments is the basis for using arithmetic in solving, say, engineering problems.

The field of research which concerns itself with referential assignments is usually called "model-theoretic semantics" or "referential semantics". The general approach is the following. Given assignments of referents for the primitive symbols of a language, specify rules of combination such that, for any given well-formed expression constructed out of those primitive

symbols, the referent for the complex expression can be identified. In the example mentioned above, the rules of combination would dictate that assigning the length of some object to the primitive symbol "8" and the height of that object to "4" implies that the complex symbol "(8 * 4)" refers to the area of the side of the object. As the example indicates, the justification for referential assignments are not always obvious.

Natural language is obviously acquired by using it to talk about and describe the world. A natural hypothesis is, then, that the acquisition of the language of arithmetic is facilitated by using it to describe the world, or some part of the world. I want to propose that this is the primary function of illustrations in the initial teaching of arithmetic: to be objects to which the young learner can apply his arithmetic, and thereby facilitate his/her acquisition of the language of arithmetic. Expressed differently, the purpose of an illustration is to provide the learner with a referential semantics for the language of arithmetic.

In summary, knowledge of the various senses of the arithmetic concepts and the ability to decide what arithmetic expressions refer to are components of arithmetic understanding. I propose that the primary function of an illustration in initial arithmetic instruction is to be a world which can easily be described by the language of arithmetic. By facilitating the application of arithmetic, an illustration facilitates the acquisition of the semantics of the arithmetic language.

Summary

The picture that emerges from this brief review is that different intellectual traditions have focussed on different aspects of what it means to understand arithmetic. There is no such thing as *the* meaning of arithmetic. Complete understanding of arithmetic presupposes knowledge of at least the following intellectual constructions:

- The algebraic laws of the number system

- The justifications or proofs of the algebraic laws
- The arithmetic procedures
- The teleological semantics of arithmetic procedures, i. e., the principles underlying the procedures, and the connection between the principles and particular procedural steps
- The language of arithmetic, its vocabulary and syntax
- The analytical semantics of the senses of the arithmetic concepts
- The referential semantics of arithmetic expressions

A complete theory of arithmetic understanding would have to contain the details of how each of these constructions is represented in the human mind, and how it is, or can be, acquired.

The hypothesis advanced here is that ***the acquisition of arithmetic understanding proceeds in the opposite direction as compared to the list above.*** In other words, arithmetic learning starts with the different senses and the referents of arithmetic symbols and expressions, then proceeds to an understanding of the computational procedures, and ends with an understanding of the number system and its algebraic properties.

The effects of understanding the language of arithmetic is, the hypothesis claims, that procedures and principles become easier to learn and to understand. Indeed, one can ask how it could be otherwise: How can one possibly understand a procedure which manipulates symbolic expressions, before one understands what those expressions mean? How can one understand a principle expressed in certain symbols, before one understands the meaning of those symbols? Understanding the language of arithmetic seems to be a prerequisite for understanding the content of arithmetic, in the same way that understanding natural language is a prerequisite for understanding history and geography.

The conclusion is that in constructing a theory of illustrations for initial instruction in arithmetic, we should focus on the language aspect of arithmetic. An illustration should be

conceived of as providing the learner with a referential semantics for the symbolic expressions, a world to which those expressions can be applied. By describing this world with the help of arithmetic symbols, the learner is prompted to induce the mathematical concepts behind those symbols.

In the following two sections, this view of illustrations will be developed in detail with respect to the arithmetic of rational numbers. The first section below distinguishes the many senses of the rational number concept. In the following section, one of those senses is selected and an illustration is developed for it.

Sense

In the teaching of rational numbers, the starting point consists of concepts, here to be called ***rational number concepts***, which have everyday origins but for which the arithmetic of rational numbers is the appropriate mathematical expression. Because of their obvious pedagogical importance, several attempts have been made to describe the set of rational number concepts. Two such attempts are summarized in the first subsection below. Each attempt captures a part of the truth, but the analytical methodology used is too weak. Reflection on what questions an analysis of rational number concepts ought to answer points to analytical semantics as the appropriate methodology. A sketch of such a semantics for rational number concepts is presented in the second subsection.

Rational number concepts: Existing proposals

Kieren (1975, pp. 102-103) lists the following interpretations of rational numbers:

1. "Rational numbers are fractions which can be compared, added, subtracted, etc.
2. Rational numbers are decimal fractions which form a natural extension (via our numeration system) to the whole numbers.
3. Rational numbers are equivalence classes of fractions. Thus, {1/2, 2/4, 3/6, . . .} and {2/3, 4/6, 6/9, . . .} are rational numbers.

4. Rational numbers are numbers of the form p/q, where p, q are integers and q /= 0. In this form, rational numbers are "ratio" numbers.

5. Rational numbers are multiplicative operators (e. g., stretchers, shrinkers, etc.).

6. Rational numbers are elements of an infinite ordered quotient field. They are numbers of the form x = p/q where x satisfies the equation qx=p.

7. Rational numbers are measures or points on a number line."

In a later article (Kieren, 1980), the list of seven subconstructs is called a "pool" of interpretations, from which five ideas "emerge": part-whole, quotients, measures, ratios, and operators.

Before commenting on this list, let us consider a second list of seven rational number concepts, proposed by Behr, Lesh, Post, and Silver (1983, pp. 99-100), who call them "subconstructs":

- The ***fractional measure*** subconstruct addresses "the question of how much there is of a quantity relative to a specified unit of that quantity". Behr et. al. (1983) proposes this as a re-formulation of the part-whole interpretation.

- The ***ratio*** subconstruct.

- The ***rate*** subconstruct, which defines a new quantity as a relationship between two other quantities. The distinguishing mark, according to Behr et. al. (1983), of a rate (as opposed to ratio) is that rates are (frequently) added, while ratios are (usually) not.

- The ***quotient*** subconstruct which interprets the rational number as the result of division operation.

- The ***linear coordinate*** subconstruct which interprets the rational number as a point on the numberline. This subconstruct emphasizes, according to Behr et. al. (1983); that (a) rational numbers is a subset of real numbers, and that (b) "properties associated with the metric topology of the rational number line such as betweenness, density, distance, and (non)completeness" (Behr, et. al., 1983, p. 100).

- The ***decimal*** subconstruct which emphasizes "properties associated with the base-ten numeration system" (Behr, et. al., 1983, p. 100).

- The ***operator*** subconstruct which interprets a rational number as a transformation.

Comparison with the list by Kieren (1975) shows that only the division, ratio, and operator constructs appear in both lists.

These lists of interpretations can be criticized both for what they contain, and for what they do not contain. For instance, to define rational numbers as equivalence classes of fractions presupposes that fractions can be defined independently of rational numbers, which seems to contradict the idea that fractions *are* rational numbers. This is a surprising move, especially in view of the fact that Kieren at the same time defines rational numbers as fractions. Furthermore, Kieren seems to equate the concept of "measurement" and the concept of "a point on the number line" which seems to me too restrictive a view of the number line. Looking at the list by Behr et. al. (1983) we see that it contains the concept of "decimals". However, one could hardly view decimal numbers as a kind of subconstruct of rational numbers, since there are numbers which can be expressed in decimal notation but which cannot be expressed as pairs of integers (i. e., all non-rational real numbers). Also, it does not seem pedagogically useful to regard decimals as a subconstruct to rational numbers. Finally, the meaning of the linear coordinate and fractional measure constructs in the Behr et. al. (1983) list escape me.

Looking at these lists from another point of view, there are several concepts which intuitively belong in the set of rational number concepts, but which do not have a place in these lists. First, the idea of a proportion e.g., "the proportion of Americans who live in private homes is close to 80%", is very pervasive in everyday life but it does not appear in either list. Second, the operation of division can be carried out in two distinct ways: by partitioning the numerator into as many equal-sized parts as specified by the denominator and interpreting the answer as the size of one of those parts; or, alternatively, by extracting from the numerator a part of the size specified by the denominator and interpret the answer as the number of such extractions which can be carried out. This distinction fails to appear in either of the lists above.

One possible reaction to these criticisms is to try to provide a better list. However, the problem lies not so much with the content of the particular lists, but with the methodology used. For instance, Kieren (1980) claims that some of the interpretations he has listed are more basic

than others, but does not provide any explanation of how one decides whether a particular interpretation is basic or not. (For instance, are rates and ratios fundamentally different, or are rates a kind of ratio?) Also, it is unclear why the lists are not longer: How does one go about finding or constructing additional interpretations? Finally, simply listing a set of concepts does not tell us anything about the relations between the concepts in the list. Proportions and ratios are clearly more alike in meaning than, say, proportions and quotients, an observation which might have pedagogical implications. The list format does not provide such information, nor any systematic way of representing it.

The next subsection begins with a brief look at some analytical devices which have been used to elucidate the meanings of concepts, and then proceeds to lay the foundation for an analytical semantics for rational number concepts.

Towards an analytical semantics for rational number concepts

The proposals by Kieren (1975) and by Behr et al. (1983) reviewed in the previous subsection show that there is a set of interrelated concepts attached to the notion of rational numbers. Using a term taken from psycholinguistics, we can say that the rational number concepts form a ***semantic field***. Other examples of semantic fields are color concepts, and concepts of physical motion (Miller & Johnson-Laird, 1976). The concepts in a semantic field are more like each other than they are like concepts outside the field, but within the field they contrast with each other. A theory of a particular semantic field should assign a structure to that field, a structure which explains or brings out the meaning and the semantic relations between the concepts in the field. The problem of how to do this has been approached in different ways by different scholars. For purposes of the present essay, I use the term ***analytical semantics*** to refer to any such effort.

One technique of analytical semantics is to describe concepts in terms of their ***semantic primitives*** (or "features" or "components"). Similarity of meaning is then understood in terms

of shared primitives. A list of semantic primitives, and the analysis of a set of concepts in terms of them, is usually known as a "decompositional semantics" for that set of concepts. If the semantic components are procedures, so that the meaning of each concept correspond to a complex procedure composed out a set of primitive procedures, then we have an example of a "procedural semantics". See, e.g., Miller & Johnson-Laird, (1976). A much older device for exhibiting the relations between concepts is to organize them in a ***taxonomy***, i. e., a tree-like structure that makes explicit which concepts are special cases of other concepts. Similarity of meaning is then understood in terms of distance in the tree. (In Artificial Intelligence research, taxonomies go under the somewhat peculiar name "is-a hierarchies".) Ideally, an analytical semantics should be ***generative***, i. e., it should consist of a formal machinery, a set of rules, by which the structure associated with a particular semantic field can be constructed.

In the following, I attempt to describe the structure of the field of rational number concepts. I begin with the mathematical concept of a rational number as an ordered pair of integers. The first step in the analysis is to show that the distinction between quantities and parameters generates three different interpretations of rational numbers, corresponding to the notions of "comparing", "dividing", and "changing", respectively. The hypothesis to be developed is that all rational number concepts can be generated from those three notions by introducing semantic restrictions on the numerators and the denominators.

The mathematical specification of the rational number designated by $\langle a, b \rangle$ is that number r which satisfies the equation

$$rb = a.$$

where a and b are understood to be integers. (The standard terminology is that a is called the "numerator", b the "denominator", and r the "value" of the rational number.) Given this specification, what meaning can we assign to $\langle a, b \rangle$?

Integers have two basic interpretations: (a) as values or measures of (physical) ***quantities***, e.g., the length of a table, and (b) as ***parameters*** for arithmetic operations, e. g., as the number by which some other number is to be divided. Using this distinction, we can generate three[2] basic conceptions of rational numbers:

- When both the numerator and the denominator are interpreted as quantities, then the rational number is an expression of the idea of ***comparison***. One quantity is taken "in relation to" the other quantity. Depending on how the comparison is carried out and on what quantities are being compared, different types of comparisons result, as will be developed in some detail below. In this case, the value of the rational number is to be thought of as a "comparison number".

- If the numerator is taken to be the value of a quantity, while the denominator is taken to be a parameter to the operation of partitioning then the rational number expresses the concept of ***division***. In this case, the value of the rational number is known as the quotient of the numerator with respect to the denominator. (Notice that the operation for which the denominator is a parameter operates upon the numerator.)

- If both the numerator and the denominator are taken to be parameters to operations, then the rational number as a whole becomes a composite operator. It expresses the concept of ***change***, or, more specifically, the idea of counteracting changes. (Notice that the operations represented by the numerator and the denominator operate on some other quantity, not explicitly mentioned in the rational number itself.)

Thus, there are three basic views of rational numbers: as comparisons between two quantities, as results of dividing a quantity, and as counteracting changes in a quantity. I claim that these three views cannot be reduced to, or derived from, each other; they are semantically independent. Furthermore, I want to propose that all rational number concepts can be derived from these three basic notions by adding semantic constraints on the numerators and the denominators.[3]

I will distinguish between discrete and continuous quantities. Discrete quantities are represented by sets and can be quantified by counting; the number of persons in a room is a paradigmatic example. Continuous quantities can be quantified by measuring; the length of an

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[2]Strictly speaking, we can generate four combinations. However, so far I have been unable to find any meaning in the combination in which the numerator is a parameter and the denominator is a quantity.

[3]The common interpretation of rational numbers as *fractional parts* may not be covered by this hypothesis. The idea of fractional parts is, I believe, very complex. Its analysis is not included here because it would not contribute to the main purpose of the chapter. A more formal treatment of the analytical semantics of rational number concepts, including the idea of fractional parts, will be presented elsewhere.

object is a paradigmatic example. I will assume that the discrete case is more basic, because the counting operation is more basic than the measuring operation. Indeed, measuring requires knowledge of rational numbers, while counting only requires knowledge of integers.

I will now discuss comparison, division, and change in more detail.

Comparison. Suppose we have two physical quantities P and Q. What are the possible relations between them? In what different ways can they be compared or related to each other? Let us begin with two sets of countable objects, like black and white marbles in an urn, two groups of people, etc.; call them S1 and S2; call the values of these quantities P(S1) and P(S2), respectively.

We can classify the relation between P(S1) and P(S2) by looking at the relation between S1 and S2. We find four possible relations: S1 is included in S2, S_2 is included in S1, S_1 is disjoint with S2 (i. e., independent of), and S1 and S2 overlap partially.

Looking first at the situation where S1 is a subset of S2, i. e., where S1 is constrained to be a part of S2, we see that P(S1)/P(S2) implies taking the value of the subset and relating it to the value of the superset. In informal language, this concept is expressed by the phrase "x ***out of*** y", as in "winning three matches ***out of*** ten". Clearly, this concept is close to that of ***proportion***, as in "the proportion of voters who stayed home in this election was 10%". This concept answers the question "how big is the part compared to the whole?". The intuition that underlies this concept is that, say, 3 out of a hundred, is a smaller amount than, say, 3 out of ten, but that 3 out of a hundred is, in a sense, the same amount as, say; 30 out of a thousand. Notice that if we restrict S1 and S2 to be sets of events, then we generate the concept of ***probability***. Thus, a probability is a kind of proportion, which is a kind of comparison between a part and a whole, which in turn is a subspecies of comparisons in general.

Looking next at the situation where S2 is a subset of S1, we see that P(S1)/P(S2) implies comparing the superset to one of its parts or subsets. This comparison concept answers the question "How many times bigger is S1 than S2?" This concept does not have a common every day term, but call it "multiplicative comparison".[4] If we restrict P(S2) to be a standard quantity, we generate the idea of measurement. To say that something is, say, 10 meters long is to say that it is 10 times bigger than the 1-meter standard. Thus, measurement is a species of multiplicative comparisons, which is a species of comparison which is a kind of comparison between a whole and as part, which, in terms, is a species of comparisons in general.

Finally, looking at the situation where S1 and S2 are independent, taking the value of one over the value of the other implies asking "How much of S1 there is in relation to S2?", or, with discrete quantities, "How many members of S1 are there for each member of S2?" This is the ***ratio*** concept, as in "there are two girls for every boy in this class" or in "this rectangle is twice as long as it is wide". If we further restrict the denominator to be a quantity of time, we generate the concept of a ***rate***, as in "there are three accidents per weekend". In other words, a rate is a kind of ratio, which is a comparison between wholes, which, in turn, is a subspecies of comparison in general.

In summary, by considering the possible relations between two quantities, we generate three basic cases, each giving rise to a family of rational number concepts: concepts having to do with comparing a part to a whole, concepts having to do with comparing a whole to a part, and comparing two independent quantities. These correspond to the notions of "x out of y", of "x times bigger than y" , and of "x P:s for each y Q:s," respectively. (For the case of continuous quantities, the last alternative becomes "x units of P for each y units of Q".)

Division. The comparison concepts presuppose that the two integers which form a rational

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[4]Additive comparison answers the question "How much bigger is the x than y?"

number are to be interpreted as quantities. However, the notion of ***division*** stems from a different basic idea, namely that of dividing or partitioning a quantity into equal-sized parts. The numerator is then interpreted as the quantity to be partitioned, and the denominator is interpreted as the parameter to the partitioning operation, i. e., as the specification of the number of parts into which the quantity is to be partitioned. The rational number in this case answers the question "How big are the resulting parts?" Whereas the rational number in the previous case was interpreted as a "comparison number" which expresses the relationship between two quantities, in this case the value of the rational number is a quantity, namely the size of the part which results from carrying out the partitioning operation. For instance, if a rope 3 meters long is divided by 3, then the length of the part is one meter.

The idea of partitioning separates into two distinct concepts, each appropriate for a different type of quantity. For a discrete quantity, partitioning takes the form of ***sorting*** the elements in the corresponding set into separate sets. For a continuous quantity, partitioning takes the form of ***cutting*** the quantity, or the object it is associated with, into parts. The difference between these two operations is that the sorting operation cannot always be performed under the constraint of equal-sized parts, while the cutting operation is always, in principle, possible.

There is a peculiar semantic mismatch between the division concept and the standard arithmetic notation. The value of a rational number in the division interpretation refers to *one* of the parts which result from the partitioning operation. Dividing a rope into three parts gives us three shorter ropes, but the mathematical description of this operation does not express this fact. Semantically speaking, the equation

$$\frac{6}{3} = 2$$

is ill-formed, because from the point of view of the division concept one would read it as saying

that "if we have a quantity of 6 and divide it into three parts, we are left with a quantity of 2", which is incorrect or at least incomplete because it only mentions one quantity of size 2; fails to mention the fate of the other two parts created by the division. From the point of view of division, a more perspicuous notation would be

$$\frac{6}{2} = 3*2$$

which has the semantically well-formed reading "a quantity of size 6 divided into 3 parts yields 3 quantities of size 2", but which is inconsistent with the standard interpretation of the arithmetic notation.

One resolution to this anomaly is to re-interpret the concept of division as containing two semantic components, the idea of partitioning and the idea of ***selecting***, namely selecting one of the parts which are created by the partitioning operation. The first expression above then has the semantically well-formed reading "if we partition a quantity of size 6 into 3 parts, and ***select*** one of those parts, then that part has the size 2". The problem with this move is its arbitrariness; we are perfectly capable of carrying out sorting and cutting operations in the real world without at the same time performing selection operations; there is no obvious reason why the two should have to be inextricably joined in the mathematical description.

Another resolution is to re-interpret the division concept as answering the question "How many parts of size x does a quantity of size y contain?". This involves a radical re-interpretation of the rational number. In the new interpretation, the numerator stands for the quantity to be partitioned as before, but the denominator is an expression of the size of the desired part(s), and the value of the rational number stands for the number of such parts. In effect, the denominator and the value of the rational number, i. e., the two parameters to the partitioning operation, have "changed places" as compared to the interpretation discussed above. The concept of division

now has become synonymous with the concept of *extraction*, of taking a part and extracting it from a whole until the latter has been "emptied" or "exhausted". The value of the rational number is the number of times the extraction operation can be performed before the whole is exhausted.

The advantage of this concept is that it applies equally well to discrete and to continuous quantities. Also, the standard arithmetic notation now has a semantically well-formed reading: "if we take a quantity of size 6, and repeatedly extract parts of size 3 from it, we will be able to extract 2 such parts before the quantity is exhausted". The need to introduce the idea of selection in order to make the standard notation semantically well-formed has disappeared.

The concept of successive extraction of parts of a given size is rather different from the concept of partitioning, but it is similar to the concept of multiplicative comparison, i. e., of comparing two quantities, one of which is a subset of the first. However, in this case, the concept is an active one rather than just a relational one, and the value of the rational number is interpreted as describing how many times the extraction operation is performed, rather than as a "comparison number" which expresses the relationship between the two quantities.

In summary, the concept of division gives rise to a subset of rational number concepts which have to do with various ways of "taking apart" a quantity. The two notions of cutting and sorting implement the notion of partitioning a given quantity into a given number of equal-sized parts for continuous and discrete quantities, respectively, while the notion of extracting a part of a given size is applicable to either. The latter is closely related to the concept of multiplicative comparison.

Change. The final possibility is to let both the numerator and the denominator be parameters of operations. The entire rational number is then conceived of as an operator "x/y" which

corresponds to the composite operation **div(mult(x, Q), y)**. Notice that the notion of an operator presupposes an operand, i. e., something for the operator to operate upon. To operate with the entire rational number is to compute what the final result is when both processes operate on the same operand.

Depending upon the interpretation of the multiplication and the division operations, respectively, we get different concepts within this family. Concepts of counteracting processes are rather common in natural science. In an electric circuit, the voltage can be seen as an operator which increases the amount of current through the circuit, and the resistance of the circuit as an operator which decreases the amount of current. Thus, the rational number <voltage>/<resistance> operates upon some constant k in order to give the current:

$$\text{Current} = k\ \text{Voltage}/\text{Resistance}$$

Clearly, formulas of this kind can be interpreted as saying that an increase in one quantity can be offset by an increase in another, such that a third quantity is kept constant. The two processes of "increasing voltage" and "increasing resistance" are counteracting processes in this sense.[5]

Summary

It is suggested that the different senses of the rational number concept constitute a semantic field in the sense in which this construct is used in the study of semantics of rational numbers. Two previous attempts to describe this semantic field were criticized and an alternative analysis proposed. By starting with the concept of an ordered pair of integers and introducing semantic constraints on the members of the pair, we generated three basic senses for rational numbers, each of which subdivides further into more semantically constrained concepts, etc. The three

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[5] It is noteworthy that both "voltage" and resistance" are quantities. Thus, we can interpret the rational number <voltage>/<resistance> as a comparison between two quantities as well. It seems plausible that this way of interpreting the rational number is appropriate for stable circuits, while the composite operation interpretation is appropriate for changing circuits.

basic senses are as ***comparisons*** between quantities, as the result of the ***division*** of quantities, and as ***counteracting changes***. The comparison sense divides further into concepts like proportion, ratio, and multiplicative comparison, depending upon the relation between the quantities being compared. Further distinctions, such as that between ratio and rate, can be introduced by further specifying the quantities involved. The division sense comprises the ideas of partitioning and of extracting; the former splits into cutting and sorting depending upon the nature of the quantities involved. The change sense includes many different kinds of change, depending on how the numerator and the denominator are interpreted, respectively. The semantic distance between any two senses can be specified in terms of the choices made in their derivation. See Figure 4.

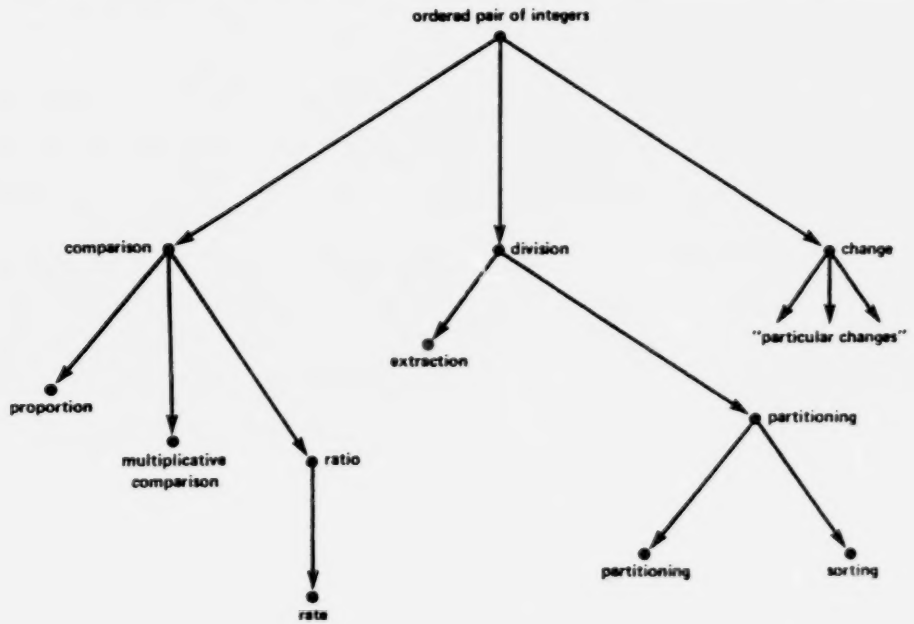

Figure 4. A hypothesis about the semantic structure of the field of rational number concepts. (Notice that the semantic distinctions which generate the branches at each node in the tree have not been specified. See text.)

The present analysis only goes part of the way towards the goal of constructing an analytical

semantics for the field of rational number concepts. The treatment is informal and there are a number of questions not answered by the analysis in its current form. For instance, it does not answer the question of what the boundaries are on this semantic field. Is there an infinitude of (possible) rational number concepts, or is it a bounded set? Also, the present treatment does not provide a metric for the semantic distance between any two concepts in the field.

However, in spite of the informality of the treatment, it supports the conclusion that a person cannot be said to understand the concept of rational number unless he or she has knowledge of the various senses of that concept. Thus, in constructing a referential semantics - an illustration - for rational numbers, we must begin by selecting a particular sense to be illustrated. This conclusion will find immediate application in the next section.

Reference

A concept has many kinds of relations to other concepts, most notably relations of similarity and difference of meaning. Analytical semantics deal with these, as described in the previous section. However, concepts are used to understand the world and so have relations to objects and events in that world, usually called relations of reference. Referential semantics deal with these.

A referential semantics for a language specify how the expressions in that language refer to the world outside the language, i. e., which object or state of affairs each expression symbolizes. Such a semantics consists of three components. First, there is a vocabulary, a set of primitive symbols. Second, there are rules for which objects or states of affairs the primitive symbols refer to. Finally, there are rules of combination which specify how to compute what the referent for a complex expression is, given the referents for the primitive expressions in it. In short, a referential semantics for a language specifies how that language describes the world.[6]

The purpose of this section is to give an example of a referential semantics for rational

[6]See Anderson (1976), Chap 7, for a different use of a referential semantics in a psychological context.

numbers. The general method used here is as follows:

- Select a rational number ***concept*** - a sense - which is to be the basis of the illustration.
- Create a "world" by specifying what kind of objects appear in it and what kind of properties they have, i. e., by specifying its ***ontology***.
- Describe the kinds of actions that will be allowed on these objects. We might call this the ***action space*** of the world.
- Define the ***referential mapping*** from the primitive symbols in the arithmetic language to the objects and their properties.
- Postulate ***rules of combination*** which specifies how the referent of a complex expression can be computed, given the referents of the primitive symbols in it.

The next section carries out these five steps with respect to a novel illustration for the ratio concept.

A referential semantics for the ratio concept

Selecting a sense. I have chosen the concept of ratio for the present example. As discussed in the previous section, the concept of ratio is a comparative concept, involving a comparison of the type "how much is there of P for each unit of Q" for some quantities P and Q. In this illustration, we are dealing with a special case of this concept, in which the comparison is carried out between two quantitative properties of one and the same object:

$$\frac{\langle \text{measure of object O on dimension f} \rangle}{\langle \text{measure of object O on dimension g} \rangle}$$

In this interpretation the value of the rational number is to be interpreted as an expression of how big the object O is on dimension f in relation to how big it is on dimension g. The particular dimensions used are the height and the width of a rectangle.

An ontology for a rectangle world. In this illustration, there exists:

- A fixed unit of length (which is not named)
- An infinite number of rectangles. Each rectangle is a certain number of length units long and an certain number of length units high. The units are marked off along the base and along the (leftmost) side with what we might call ***unit markers***, e. g. little dashes
- Diagonals, which are straight lines from the lower, left-hand corner of a rectangle to the upper, right-corner
- Dividers, which are horizontal lines which cut a rectangle into two parts (i. e., into two rectangles); dividers are always aligned with one of the unit markers

The action space of the rectangle world. In the rectangle world, the following actions are possible:

- An operation which ***creates*** a rectangle with a specified length and height
- An operation which ***throws away*** a rectangle
- An operation which ***moves*** a rectangle around
- An operation which ***duplicates*** a given rectangle
- An operation which ***cuts*** a given rectangle horizontally at the height of any of the left-hand side unit markers by drawing a divider across the rectangle (and thus creates two rectangles)
- An operation which ***draws*** a diagonal in a rectangle
- An operation which ***superimposes*** one rectangle over another
- An operation which ***stacks*** one rectangle on top of another
- An operation which ***unstacks*** one rectangle from the top of a stack of them, i. e., it takes the top rectangle in the stack and puts it "down" beside the stack

The set of situations which belong to the Rectangle World can be defined generatively as all rectangle configurations which can be reached by starting with an empty world and applying any sequence of the above actions.

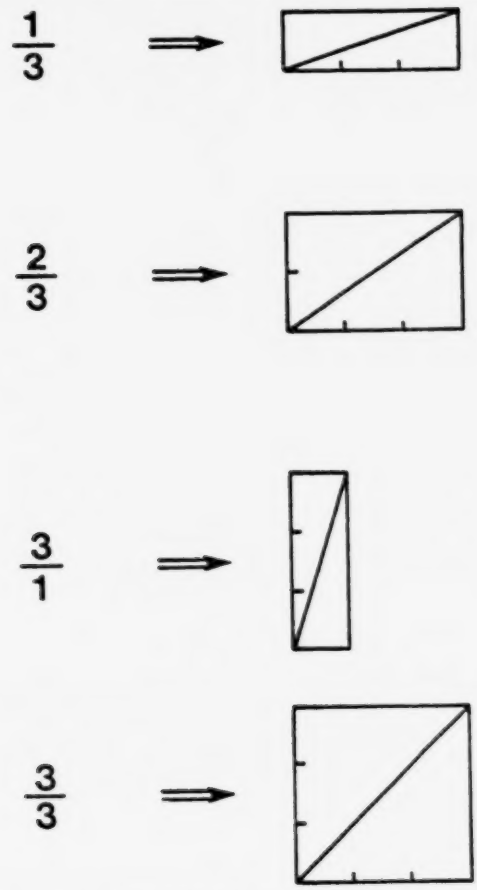

Figure 5. The referential mapping between arithmetic symbols and the Rectangle World.

The referential mapping of primitive expressions. The rational number symbols refer to the Rectangle World in the following way:

- A numerator refers to the height of a rectangle, expressed in terms of the unnamed unit of length.
- A denominator refers to the width of a rectangle.
- The value of a rational number (therefore) refers to the diagonal of a rectangle, or, more precisely, to its angle of incline, its steepness.
- An integer refers to a set of rectangles.

See Figure 5 for some examples.

Rules of combination for complex expressions. There are, of course, a very large number of

mathematical expressions which one could provide combination rules for. I will only specify rules for expressions of the following two types: "a/b" and "(x + y)", where a and b are integers, and x and y are rational numbers, i. e., for the symbols for rational numbers and for the expressions which refer to the sum and difference of two rational numbers.

The rule for constructing the referent of a rational number <a, b> are obvious from the above: Create a rectangle with height a and width b, and draw its diagonal.

The rules for finding the referent of the sum of two rational numbers is as follows: Construct the rectangles corresponding to the two rational numbers, and form a new rectangle by putting one rectangle on top of the other, thereby forming a new rectangle. Draw the diagonal in the new rectangle. (Notice that this presupposes that the bases of the two rectangles are the same, i. e., one can only add rational numbers with equal denominators in this world.)

The procedure for constructing the referent for the difference between two rational numbers is exactly the inverse of the procedure for finding the referent for the sum: Take a rectangle, divide it horizontally into two rectangles, take off the top rectangle, and throw it away. Draw the diagonal in the bottom rectangle.

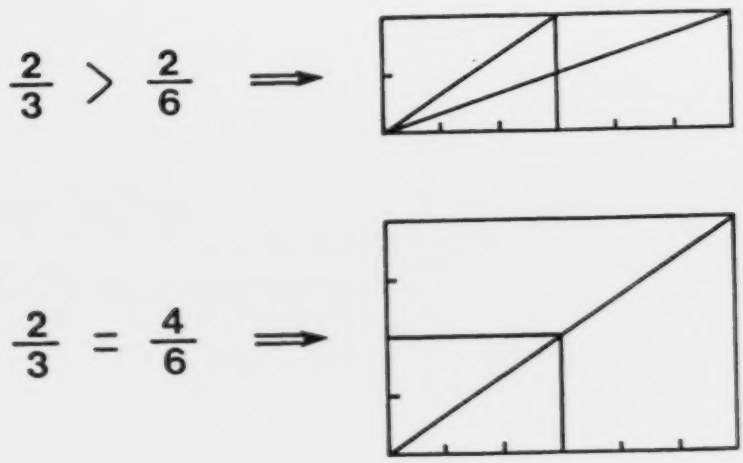

Figure 6. The relations "larger than" and "equivalent to" as illustrated in the Rectangle World.

Evaluation of the Rectangle Illustration

The referential mapping between rational numbers and the rectangles sets up a correspondence between the two worlds. The instructional value of this correspondence, its pedagogical power, so to speak, is a function of how many ideas and concepts related to rational numbers that it can illustrate. The following properties of the rectangle world are worth noticing (see Figure 7):

- Unlike most illustrations, the Rectangle World provides distinct referents for (a) the numerator, (b) the denominator, ***and*** (c) the value of the rational number.
- Some classes of rational numbers have distinct representations in this world.
  - The class of all rational numbers equivalent to some specific rational number is the class of all rectangles with coinciding diagonals.
  - The class of unit-valued rational numbers is the class of squares.
  - Rational numbers with value smaller than unity will be wider than they are high; those with values greater than unity will be higher than they are wide - a very distinct visual difference.

- It is as natural to have numerators larger than the denominator in this illustration as it is to have numerators smaller than the denominator, as should be the case in an illustration for the ratio concept. (Many other illustrations for rational numbers have difficulties with this point.)

- Comparisons between rational numbers reduces to a perceptual comparison between the "steepness" of straight lines. If one diagonal is "steeper" than another, then the corresponding rational number is greater than the other.

However, the rectangle illustration also contains the following conceptual anomaly: Rational numbers with zero numerator are as "impossible" or "illegal" as rational numbers with zero denominator. In both cases, the rectangle "disappears" or is impossible to create, because it will have zero magnitude on either height or width. The dramatic mathematical difference between a zero numerator and a zero denominator has no explanation or representation within the Rectangle World.

Also, there are many types of arithmetic expressions which do not have referents in this world, i. e., no combination rules can be provided for them. They include the sum and the quotient of two rational numbers.[7]

[7]There is no proof that such rules do not exist. My colleague, Pearla Nesher, has indeed tried to convince me that the Rectangle World *can* illustrate the entire arithmetic of rational numbers, but so far I have been unable to construct natural representations for multiplication and division which are consistent with the referential mapping of this world as I have defined it.

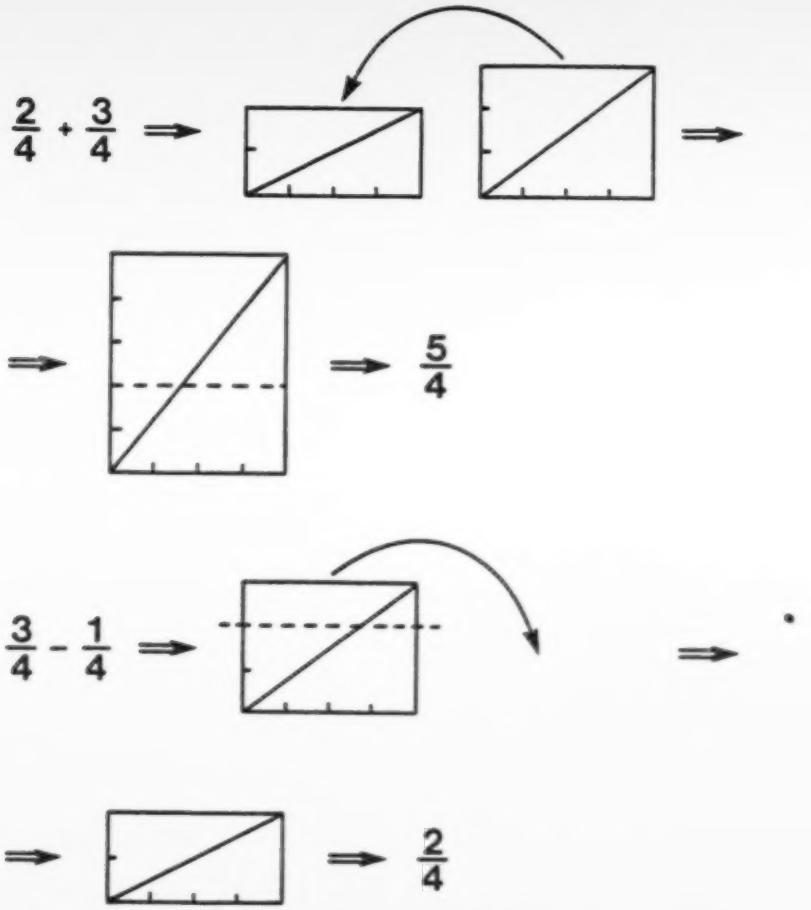

Figure 7. How to find sums and differences in the Rectangle World.

The observations of the proceeding paragraph are, I believe, instances of a more general principle. In designing an illustration, it is tempting to believe that we can illustrate any arbitrary part of arithmetic. However, the analyses I have carried out in connection with the Rectangle World and several other illustration worlds leads me to the conjecture that ***there is no illustration which is completely isomorphic with the arithmetic of rational numbers.*** One cannot define an ontology and a referential mapping between the arithmetic symbols and that ontology such that one can create combination rules for every type of arithmetic expression consistent with that referential mapping. The structure of rational number arithmetic is such that objects and events in the physical world are inherently incapable of mirroring (all of) that structure. This conclusion, if correct, has important implications for the design of illustrations.

A Theory of Illustration Design

It is a commonly held pedagogical goal to teach arithmetic in such a way that learners understand arithmetic concepts, principles, and procedures. The obvious desirability of the goal has obscured the unobvious nature of the goal: What does it mean to understand arithmetic? According to the analysis presented here, arithmetic understanding involves knowledge of at least the following intellectual constructions:

- The language of arithmetic, i. e., the vocabulary of arithmetic symbols and the syntax of arithmetic formulas
- The analytical semantics of arithmetic, i. e., the different senses which can be expressed by arithmetic symbols and how they relate to each other
- The referential semantics of arithmetic, i. e., the possible mappings between arithmetical symbols and the world
- The properties of the mathematical objects involved
- The computational procedures of arithmetic, i. e., the algorithms for constructing the sum, the difference, the product, or the quotient of various kinds of numbers
- The teleological semantics of arithmetic, i. e., the principles which shape the computational procedures and the connections between those principles and the steps in the procedures
- The abstract specification of the computational procedures, i. e., the algebraic laws which define the arithmetic functions
- Proofs of the algebraic laws, i.e., justifications or reasons for why one should believe in the truth of those laws

In short, there is no such thing as *the* meaning of arithmetic. Arithmetic is not a well-specified body of knowledge, but involves all of the intellectual constructions mentioned above, each of which can be known and mastered in varying degrees and to varying extents.

The fundamental issue in illustration design is which of the intellectual constructions mentioned above is supposed to be clarified by a particular illustration. I argued that the main task of an illustration for initial instruction in arithmetic should not be conceived of as illustrating the algebraic laws of the number system, because understanding of arithmetic

precedes knowledge of those laws. Also, I argued that the main task of an illustration for initial instruction in arithmetic is not to provide an isomorphic procedure to the arithmetic procedure, because (a) isomorphism is hard to achieve, (b) if the isomorphism is imperfect, there is no reason to expect transfer from the illustrative procedure to the target procedure, and (c) is the isomorphism is perfect, there is no reason to expect the illustrative procedure to be easier to learn.

The main hypothesis proposed here is that the primary function of an illustration in the initial teaching of arithmetic is to help the young learner to understand the language of mathematics by providing a referential semantics for it. Having a referential semantics facilitates acquiring the arithmetic concepts in several ways, perhaps primarily by giving the learner something to abstract from. Understanding the language of arithmetic facilitates, in turn, the acquisition of procedures and principles which deal with expressions in that language.

The second hypothesis I want to propose is that an illustration intended to function as a referential semantics for the language of arithmetic should be constructed in the following way:

- Identify the various senses for the mathematical concept to be illustrated. If there are several, select one.
- Construct an illustration world. This involves the following steps:
    - Describe the primitive objects of the world and their properties (the ontology).
    - Describe the possible actions (the action space).

  The illustration world is the set of all configurations (situations) which can be generated by applying the actions to the primitive objects.
- Construct a referential mapping between the mathematical notation and the illustration world. This involves two steps:
    - Set up the mapping between primitive symbols and the world.
    - Define combination rules which specify how the referent for a complex symbol can be computed, given the referents for the primitive symbols.

  The goal is that every configuration in the illustration world should have a meaningful

mathematical description, under the particular mapping defined.

- Evaluate the illustration. This involves asking the following questions:
  - Which concepts, procedures, and principles have (natural) representations in the illustration world?
  - Which anomalies are there in the illustration? I. e., which incorrect conclusions about arithmetic does the illustration sanction or suggest?

This methodology was applied to the concept of rational number. The analytical semantics presented shows that there is a large set of different interpretations of rational numbers. Selecting the ratio sense, I proceeded to construct a referential semantics for it, namely the Rectangle World. Analysis of the Rectangle World revealed several properties which, I conjecture, are typical of illustrations in general. Some arithmetic concepts turned out to have natural characterizations within the illustration world. For instance "unit-valued rational numbers" correspond to "squares". As a second instance, "equivalent rational numbers" correspond to "rectangles with coinciding diagonals". Also, some rules of rational number arithmetic are explained in a natural way. For instance, the addition of rational numbers (i.e., the stacking of rectangles) requires equal denominators (i.e., rectangles with equal bases).

However, the illustration also contained anomalies and gaps. The referential mapping used in the design of the rectangle illustration implies that rational numbers with zero numerator are as "illegal" or "impossible" as rational numbers with zero denominators. The mathematical difference between them has no explanation or representation in the rectangle illustration. Furthermore, the operations of multiplying and dividing rational numbers have no natural representation in this illustration world. The illustration is radically incomplete.

This prescriptive theory of illustration design is currently being used in the construction of several computer-based instructional systems for a variety of topics in elementary mathematics. If the theory is good, illustrations designed in accordance with it should prove to be pedagogically

powerful. Thus, empirical evaluation of those instructional systems will include an empirical evaluation of the present theory.

Discussion

The theory of illustration design presented here has been proposed in the context of initial instruction in arithmetic. How useful would the theory be outside this domain? The theory will be useful for designing illustrations in some other domain or topic to the extent that the considerations which lead to it hold in that other domain as well. First, there must be some significant notation introduced which is specific to the topic. Second, understanding of this notation must be crucial to the procedures and principles which belong to the domain. Clearly, many areas of mathematics fulfill these criteria, e. g., calculus[8]. However, there are also areas of mathematics in which no new symbolic notations are introduced; instead, well-known symbols are used for new purposes, e. g., trigonometry. In such cases, it is probably not useful to conceive of illustrations as providing the learner with a referential semantics.

The theory of illustration design presented here deals, in principle, with illustrations in general, regardless of the medium in which the illustration is presented. However, the theory was constructed in the context of the design of systems for intelligent computer-aided instruction, and there are some aspects of the theory which favour the computer as the appropriate delivery vehicle. First, the concept of referential mapping is central to the theory (see previous sections). A computer can be programmed to recognize the referential mapping between the mathematical notation and the illustration world, i. e., to keep track of which number on the screen was introduced as a description of which object. Such reification of the referential mapping has a number of pedagogical uses (see below).

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[8]The teaching of calculus has already settled on a very useful illustration world of its own, namely the function graph world.

Second, the theory asserts that illustration worlds should be defined generatively (see previous sections). When an illustration world is defined on a computer, the generative character of the definition translates into exploratory freedom for the learner. The learner can literally discover how mathematics describes the world by creating any arbitrary configuration in the illustration world, and then have the computer tell him what the correct mathematical description is.

There are other, less theoretically motivated, advantages of computer-based illustrations as well. Computerized illustrations are more practical in a number of ways than physical embodiments. We can program the computer to do the busy-work connected with the illustration world (e. g. coloring a pie-chart) for the child, instantaneously and correctly, thus eliminating the time otherwise spent in managing activities with little pedagogical value. Also, it is easier to switch back and forth between alternative illustrations on a computer; no time has to be spent packing away the equipment associated with one type of illustration (e. g. bundles of sticks with rubber bands) in order to make room for another (e. g. felt strips).

In short, the computer brings both new principal possibilities to illustration use and increases the practicality of teaching with the help of illustrations. It is noteworthy that these advantages of the computer as a delivery vehicle for pedagogical illustrations are not dependent upon the use of advanced artificial intelligence techniques.

One issue which has not been touched upon in this chapter is how illustrations should be used in teaching. What pedagogical use should we make of them, once they are designed? VanLehn and Brown (1980) suggest that one should present a sequence of illustrations which successively come closer and closer to the content one wants to teach. Resnick (1982) has suggested that one can help the learner generalize by having several illustrations for the same piece of mathematics. Another natural idea is to try to use illustrations to prevent errors and misconceptions from developing or remaining. For instance, one could present an illustration to the learner as an

impasse-breaking device: whenever the learner hesitates about what step to take next in a computation, he/she could be encouraged to work out the corresponding problem in an illustration, to see if that helps him understand what he ought to do in the original problem. Another interesting idea is the concept of *yoking* (Resnick and Omanson, in press) which suggests that one should hook one illustration either to another illustration or to the arithmetic symbols in such a manner that whenever the learner changes one of them, the corresponding change in the other occurs automatically. The issue of how to teach with the help of interactive illustrations is unexplored and wide open for experimentation.

The ultimate goal of using illustration in the teaching of rational numbers is, of course, that the learner should come to realize that the mathematics of the various rational concepts is the same, i. e., that the addition of proportions follows exactly the same laws as the addition of quotients, that the multiplication of a ratio with an integer yields exactly the same value as the multiplication of a fraction with an integer, etc. The laws and principles of rational numbers are insensitive to how the user of rational numbers thinks about them. Somewhat paradoxically, then, the ultimate purpose of assigning meaning to arithmetic symbols is to enable the learner to come to understand that the meaning is not important, indeed, that the meaning has to be ignored if the full generality of the mathematical concept of rational number is to be understood. The paradox of arithmetic teaching is that the teacher cannot communicate this directly, cannot simply tell the learner that proportions, ratios, and quotients are added in the same way; if he/she does, then the knowledge is useless. The generalization has to be constructed anew by each generation of learners. The ultimate criterion for a good illustration is one which facilitates this discovery, i. e., one which makes itself unnecessary as soon as possible.

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Notes

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